Why it is important for in-service elementary mathematics teachers to understand the equality \(0.999\ldots = 1\)

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**Abstract**

Researchers conducted semi-structured interviews with in-service fifth grade teachers. The purpose of these interviews was to examine teachers’ reactions to arguments that \(0.999\ldots = 1\). Previously reported results indicate that some pre-service elementary school teachers possess misunderstandings about mathematical issues related to decimals with single repeating digits. This research investigates whether some in-service teachers possess misunderstandings about mathematical issues related to \(0.999\ldots\). This paper reports on one instance of a teacher whose responses indicate that the teacher’s sense of number and sense of measurement are intertwined, resulting in fragile understanding of repeating decimals. These data present evidence that teachers continue to develop repeated decimal understandings and misunderstandings throughout their careers, and that the curriculum, everyday experience, and perceptions of student learning combine to form or reinforce these understandings. Because decimals with a single repeating digit (e.g. \(0.333\ldots\) and \(0.666\ldots\)) are an integral part of the elementary mathematics curriculum, we argue that it is important that in-service elementary mathematics teachers have a clear understanding of concepts related to the concept of infinity as they emerge through the study of the equality \(0.999\ldots = 1\).

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1. Introduction

This study investigates in-service elementary mathematics teachers’ understandings of concepts related to decimals with single repeating digits. We conducted semi-structured interviews with in-service fifth grade teachers. The purpose of these interviews was to examine the teachers’ reactions to arguments that \(0.999\ldots = 1\). Previously reported results indicate that some pre-service elementary school teachers possess misunderstandings about mathematical issues related to decimals with single repeating digits. The fact that \(0.999\ldots\) is equal to 1 (see Richman, 1999 for intuitive arguments for this equality) is counterintuitive to some pre-service elementary school teachers (Burroughs & Yopp, 2010). This research investigates whether some in-service teachers possess misunderstandings about mathematical issues related to \(0.999\ldots\) and whether discussions about this equality have the potential to expose other misconceptions about repeating decimals.

1.1. Literature review

Questions about how individuals understand the equality \(0.999\ldots = 1\) have largely been confined to studies of mathematics students in calculus or real analysis. For a thorough review of the literature on the broad topic of student understanding...
of concepts related to the concept of infinity, see Dubinsky, Weller, McDonald, and Brown (2005) and the references cited therein. Tall and Schwarzenberger (1978), though not the first to conduct studies in this area, found that the majority of students in a first-year university course thought that .999... was less than one. The researchers posited that students were in a great deal of conflict over repeating decimals, sequences, and limits. Others in the mathematics education community examined the trend further, and found that difficulties with limits often stem from the mistaken belief that there are infinitesimally small gaps between real numbers (Sierpinska, 1987).

In later research, Tall (1994) addressed the difficulties students have understanding the infinite decimal representation of both irrational and repeating rational decimal expansions. He noted that students face a conflict between a desire for precision and the practical need for decimals. For example, a student with a poor sense of irrational numbers may see no harm in approximating an irrational number with a finite decimal expansion during computations.

Tall and Schwarzenberger (1978) theorized that if teachers held misconceptions regarding infinite decimals and sequences, and communicated this unease to students, then this could be one use of the difficulties students have with these related concepts. Recently some researchers have been interested in pre-service teachers' conceptions, including Burroughs and Yopp (2010) and Weller, Arnon, and Dubinsky (2009). Weller et al. (2009) report some success in improving pre-service teachers' understanding of repeating decimals, and in particular the relationship between .999... and 1, through a specially designed unit on repeating decimals in a number and operations course.

Left largely unexplored is how in-service teachers understand the concept of decimals with a single repeating digit. This leads to the research questions at the foundation of the present study. Evidence among university students and pre-service teachers shows incomplete understanding of .999... = 1. But elementary school teachers who teach facts about repeating decimals interact with these ideas as part of the curriculum. How does interaction with the curriculum influence teachers’ understanding of decimals with a single repeating digit?

1.2. Research questions

We define cognitive disequilibrium as the state of confusion that results when individuals are faced with information that conflicts with their prior held belief or understanding. By asking in-service elementary teachers questions designed to invoke cognitive disequilibrium surrounding decimals with a single repeating digit, we hoped to answer the following questions.

1. What does the cognitive disequilibrium resulting from discussion about point nine repeated reveal about in-service teachers' understanding of repeating decimals with a single repeating digit?
2. What types of conflict resolution strategies are invoked by participants when they are confronted with cognitive disequilibrium about the relationship between point nine repeated and one?
3. When faced with cognitive disequilibrium about the relationship between point nine repeated and one, what do teachers reveal about misconceptions they hold and ways they might transmit these misconceptions to their students?

2. Methodology

2.1. Methodological tool

An interview protocol (Appendix A) was designed to gather data about participants' knowledge of whole numbers, fractions, and repeating decimals and the relationships among these concepts. The protocol also asked about participants' own knowledge of relationships between the operations of multiplication and division and about their beliefs about .999... = 1. This part of the interview gave the interviewer insight into a participant's real number and real number operation understanding and set the stage for the rest of the interview by invoking mathematical thinking that would be used to lead the participant into cognitive disequilibrium.

The researchers were aware that teachers might intertwine their own mathematical understandings with their knowledge of their students and the curriculum they teach (Leikin & Levav-Waynberg, 2007). The semi-structured nature of the interviews allowed the teachers to give responses focused on their own mathematics knowledge and their knowledge of teaching. The structure also allowed the interviewer to ask follow-up questions when the teacher focused solely on his or her teaching or students' understanding. It was not our intent to classify participants' responses as solely within the domain of their own content knowledge or their teaching knowledge. However, we are cautious in our reporting and we indicate when the teacher was referring strictly to his or her teaching and when the teacher was referring strictly to his or her own understandings and beliefs.

Participants' own responses to these questions were used to present the participant with questions designed to elicit cognitive disequilibrium about the relationship between .999... and 1. This was done by assisting the participant in developing three arguments, one based on multiplication (lines A10–A12), one based on addition (lines A13–A15), and one based on position on the number line (lines A19). These arguments presented evidence supported by the participants' own assertions in earlier parts of the interview to show .999... = 1.

After the interviews were transcribed, each researcher examined the transcripts for themes. With each interview, the researchers followed an open data exploration and theoretical sampling technique (Corbin & Strauss, 2008), which
encourages the gathering of data based on evolving concepts, to elucidate the varying properties and dimensions of repeating decimal conceptions among the in-service teachers participating.

2.2. Participants

This research was conducted in the United States. Invitations to respond to an online questionnaire were emailed to all fifth grade teachers teaching at public schools located in the same county as the researchers, resulting in a pool of potential participants who were geographically convenient to the researchers. The questionnaire was used to determine respondents' beliefs about repeating decimals and their years of teaching experience. The researchers' selection process was to choose teachers who demonstrated misconceptions about repeating decimals, particularly .999... The first three respondents to the questionnaire filled the researchers' selection requirements and were selected as the participants in the study, based on the variation in the length of their teaching experience as well as their stated beliefs about decimals with single repeating digits. Their answers all indicated they probably didn't believe that .999... equals 1. The interviews were conducted in the ninth month of a 10-month school year. The participants were assigned pseudonyms.

While all three teachers provided data that contributes to answering the research questions, data from the participant with over twenty years of teaching experience, Maude, provided the opportunity for rich analysis and form the basis for this paper. Consequently, this article focuses primarily on the responses given by Maude. The other two participants' responses are included only when they offer insights into responses given by Maude.

Maude has over 20 years of teaching experience and a master's degree in literacy. Her mathematics content background includes coursework in mathematics for elementary school teachers, and no calculus. Tara has one year of teaching experience and a BS degree in elementary education. Jeffrey has 4 years of teaching experience, but this is his first year teaching mathematics. He is enrolled in a master's program in curriculum and instruction. Both have backgrounds in mathematics coursework comparable to Maude's.

2.3. Theoretical framework

Placing subjects in cognitive disequilibrium or cognitive conflict is a common teaching and research method (e.g. Hadas, Hershkowitz, & Schwarz, 2000; Tirosch & Graeber, 1990; Zaslavsky, 2005). Uses of this strategy appear to stem from earlier writing from authors such as Dewey (1933), Festinger (1957), and Piaget (1985). Dewey's (1933) notion of reflective thinking suggests individuals in a state of doubt or hesitation are in an act of searching or inquiring to find material to resolve the doubt and settle and dispose of the perplexity. Festinger's (1957) theory of "cognitive dissonance" suggests discomfort caused by logical inconsistency or contradiction motivates an individual to modify beliefs and bring them in closer correspondence with reality. Piaget's (1985) equilibration theory suggests a process of equilibration can be stimulated by conflict or disequilibrium between an individual's schemas and external realities or even differences among schemas within the individual. As a result, as Behr and Harel (1995) assert, conflict resolution is often accomplished by the application of erroneous procedures. Thus, misconception awareness is not sufficient for better conceptual understanding and improved performance (Flavell, 1977).

For the present work, cognitive conflict or disequilibrium is not used as a teaching and learning tool but instead as an inquiry tool. Like Tirosch and Graeber (1990), this research took the perspective that teachers faced with inconsistency between their stated misconceptions would reveal much about "the resiliency of, sources of support for, the misconception" (p. 99). Also, like Tirosch and Graeber (1990, p. 99), "we felt we might also learn more about which arguments were more influential on teachers' thinking or resolve the misconception."

2.4. Data

The full transcript of all three interviews is available from the authors. The transcript from participant Maude is included in the appendix (Appendix B). In what follows, we summarize Maude's interview and include quotations from her interview in subsequent sections. Quotations from Jeffrey and Tara are included when they offer additional insight or strength to the concepts that emerged.

Maude's interview began with questions about her background with college level mathematics courses. When the formal interview began, the interviewer asked "What are the relationships between decimals and fractions?" Maude immediately responded to clarify the intent of the question, asking "How would I teach that?" and the interviewer replied, "Either one, how you understand it or how you would teach it." The question and response set the tone for the remainder of the interview, where Maude included both information about what she understands of the mathematics and how she explains this mathematics to schoolchildren.

Maude was asked about the relationships between decimals and fractions, decimals and whole numbers, and whole numbers and fractions. In response to these questions she indicated that decimals and fractions are the same thing but used in different ways, and that decimals are part of a whole. She did not discuss the decimal representation of whole numbers until prompted, and then she said a whole number has an invisible decimal point "hiding behind it." She discussed fraction and whole number representations of the same number, and how improper fractions include a whole number and a fractional part. She indicated that she only discussed with students the representation of a whole number like 5 as "five over one" when she was multiplying or dividing fractions.
The interviewer asked Maude about the relationships between multiplication and division. She responded with examples of multiplication as repeated groups of the same number and division as fair division models. When asked “How would you write six divided by two equals three as a multiplication problem?” She responded, “I might have to think about that a minute. . . . I’d put three times two equals six.” When shown the division problem, written out with the standard long division algorithm, twenty five divided by five equals one hundred twenty five, she responded that she would show twenty five times five, using the example of M&M’s. “Twenty five piles of M&M’s with five M&M’s each, [or] five piles of M&M’s with twenty five M&M’s in each.”

Then she was asked about fractions with repeating decimal equivalents. She indicated that one third and two thirds are the fractions with repeating decimal equivalents that she requires students to know. She was asked to describe the relationship between one third and point three repeating, and she indicated that she shows they are the same by using long division. She indicated that she would show their equality by writing \(1/3 = .33\ldots\), and for equality of two thirds and its decimal equivalent she would write \(2/3 = .67\).

At this point in the interview she was asked about point nine repeating and one. The interviewer asked, “Here’s a question to make you think. Some people think that one is equal to point nine repeating and some people don’t. I’d like to hear what you think about it and why.”

“I don’t think it’s equal. I don’t think its equal because, uh, I think that would be confusing to kids, to say that 99 cents can be rounded up to a dollar. You know, just right to money is immediately what I would think of. Umm. But definitely when you get to more scientific things I can see where that might be seen that way, you know, in different high level sciences, possibly, but, even then you’d think they’d want to be more particular about the size or the number. So I don’t ever think of point nine nine repeating as one. I always think of it as less than one. I think it’s the ultimate number to show kids that even this isn’t one whole. There’s a teeny bit missing. This is good for me because I never think I’m very opinionated. And then, whoa!”

The interviewer then led Maude through an argument using long division showing that point nine repeating is equal to one. “Before you showed that multiplication and long division are related, and you showed how long division connects one third and point three repeating. Can you use that to write division as multiplication?” Maude worked through her ideas, concluding that this procedure would work.

“You get point nine nine nine nine. So then I would have to talk about how umm, rounding comes into the equation. How you have to round up, round down sometimes. You know and with the zero point six six six six seven, times, one, you know you’ve rounded this up. This, you have to remember that you have a number that you’re not accounting for, that three that goes on forever. You can’t round this up to a four. This is only a three and so, you have a little sort of missing portion in two thirds and one third that when you want to multiply to check it doesn’t work so well. Yeah, but I still would not, I still don’t think I would talk to kids about how it’s like one. But maybe eventually you would get to that, because you could say yeah, point nine nine nine nine forever rounds up to one. And there you have your whole dollar. I must never have had to explain that. Because I realize now I’m struggling with how I would say that.”

The interviewer probed further, asking, “If this one tells me that three times point three repeated equals one, but this one tells me that three times three repeated is point nine repeated, which one is the true equality and which one has the rounding?” Maude responded,

“Well, you never get done with the division. It goes on forever. So unless you want to go on forever the rest of your life dividing that, you better stop, and round it. And when you multiply point three three repeating times three to check your work, you have to use logical thinking to see that the reason it doesn’t come out as one is because you never got the true number on the top.”

The interview continued with the interviewer asking Maude to add both the fractional and decimal representations of one third and two thirds. Maude responds that “three thirds is one whole,” and when asked to add the decimal representations, responds,

“I have to admit, I have to admit, I would have rounded these then [indicating .33. . . and .66. . .]. I would have rounded, I would have kept the thirty three thirty three. And I would have rounded this to sixty seven and I then would have gotten one whole.”

The interviewer asks what happens if the decimal is left in the exact form, and Maude agrees, “Yep, then you’re going to get point nine nine nine nine nine again.” The interviewer asks, “Does that show point nine repeated is equal to one, or it doesn’t?”

“No [pause]. I guess I still don’t. I think it just. I think you shouldn’t be able to take a number like point nine nine nine nine forever and turn it into one. Because what if you need nine nine nine nine hundredths of a medicine, and you give a hundred, or you give 1.0 milligrams of medicine, and it’s too much. I think questions like these are overthought, which if probably really the difference between an elementary school and a high school teacher. Because we don’t come up against theories like this being questioned like this. In high school the kids would love to get the teacher off track with that theory. I can see that. I guess eventually you might have to cave in and say that point nine nine nine
nine equals one, but only for the fact that you don't have the rest of your life, you know, to add all those repeated numbers. Which is what I always tell the kids, for the rest of your life you would be writing threes and you'd never get done. And it's such a small portion, you know you could say that, after ten thousand places or so, it's such a small portion that it's irrelevant so you could round up to one. I suppose you could look at it that way."

The interviewer probes further, asking, "Okay, so, is this a true equality then? One third equals point three repeated?" And Maude indicates that they are not equal.

"Well then you could say not true, because it never comes out even. It never ends, right, I mean, we always talk about those being equal. But uh, well, again, yeah, if you took it out forever then it would be. But we wouldn't be there to see the end, you know, I mean, but yeah, that would be a true equality, but when we round it then, no, it's not a true equality any more."

The interviewer then pursued a final approach to understanding the relationship between point nine repeated and one, by discussing the position on the number line, asking, "So if we do a number line, and place zero, one and two. If we could measure it exactly, where would you place 'a third' on the number line?"

"Okay, okay, so if I have one third and two thirds, you know, it would be pretty easy to say, I would tell the kids that's the same as point three three repeating, point six six repeating."

The interviewer clarifies, "So that's where you would put point three three repeating, right on top? Exactly on that spot?" and Maude explains,

"I would. I would put it right on the same number of line, for this age children [fifth graders]. But, not if I was teaching older children. I might show that that [indicating the decimal .33...] is just a smidgen bigger. But then yeah, if you have that [the decimal .33...] a smidgen bigger and that two thirds [as a decimal] a smidgen bigger, when you turn it to a decimal, then it's finally going to get up to one. I see that point."

Then Maude tells the interviewer that to plot point nine repeated, she would, "put it as close as I could to one without being on the one." The interview concludes with the interviewer recapping Maude's responses to the three arguments designed to create cognitive conflict. Regarding the relationship between multiplication and division, Maude replies,

"It shows point nine nine nine is close to one and it shows that these numbers go on infinitely and it shows that for everyday purposes, you have to just know that these percentages or decimals are close enough. For most every day processes."

Regarding the addition argument, Maude responds, "They're close enough. Right, and if you, and I always talk, if you're in a higher level field of science or math, you might have to break that down more. But I still don't think I would have ever said that point nine nine nine is equal to one." The interviewer gets clarification, "Even though you would say that point three three three three three repeated is equal to one third? [Or] is what you're saying, it's close enough, or this is truly equal [indicating the equality of one third and point three repeating]?" and Maude confirms,

"It's close enough, it's not true equal, but it's close enough for what most people have to use in their everyday lives. Yep it's close enough. It doesn't, When you buy something, you know it does still catch me. When I see cash registers process things like that though when it does end up really coming down to ninety nine, you know like three candy bars for a dollar and you're like, should it really be a dollar? You know, or is it going to ring up as 99 cents?"

The informal interview ended here. Maude continued to talk with the interviewer about more general topics in her classroom.

3. Analysis

An overall analysis of Maude's transcript reveals some specific misconceptions about the difference between repeating decimals and rounded decimals. When shown the addition argument, Maude acknowledges that that the fractions one-third and two-thirds add to what she calls “one whole”; however, when asked to add the decimal representations, she states “I would have to round these then.” She proceeds by expressing point three repeated as thirty three hundredths and two thirds as sixty seven hundredths and demonstrating that these sum to one.

When the interviewer asked her to leave the decimals in exact form and address the operation, Maude agrees that this will yield “point nine nine nine,” but she continues and cites an application in measuring medicine. (“Because what if you need nine nine nine nine hundredths of a medicine, and you give a hundred, or you give 1.0 milligrams of medicine, and it's too much.”) This response reaffirms Maude’s desire to relate a numerical situation to physical units used in common measurement systems. She appears not to perceive the differences between numbers and the ability to express those numbers with commonly used measurement systems. If the commonly used measurement systems do not accommodate the number, i.e. if it is impossible to express the number with the measurement units considered, then Maude compensates by rounding the number to fit the measurement system.
Maude does give the impression that she understands that there is a difference between the rounded value and the exact value of a repeating decimal. She suggests that she teaches a dynamic or process view of repeating decimals, stating “which is what I always tell the kids, for the rest of your life you would be writing threes and you’d never get done.” She also indicates that “it’s such a small point, you know you could say that, after ten thousand places or so, it’s such a small portion that it’s irrelevant so you could round up to one.”

However, in a follow-up question asking whether she believes one third and point three repeated are truly equal, she says no “because it never comes out even.” She explains this by saying “if you took it out forever then it would be [equal]. But we wouldn’t be there to see the end.” When asked where she would indicate point three repeating and one third on the number line she acknowledged that she would plot the numbers in the same place “for this age children,” but if teaching older children “I might show that that [the decimal point three repeated] is just a smidgen bigger [than one-third].”

When asked to plot point nine repeated and one on the number line, Maude states she would “put it as close as I could to one without being on the one.” However, in a recap of earlier arguments, where Maude is asked to repeat her reaction to the addition argument, she says point nine repeated and one are “close enough” and acknowledges that “if you’re in a higher level field of science or math, you might have to break that down more, but I still don’t think I would have ever said that point nine nine nine is equal to one.” The interviewer asks a final question, if one third is truly equal to point three repeating. This leads Maude to discuss purchasing things at three for a dollar. The statement “when you buy something, you know it does still catch me” indicates Maude is still grappling with these relationships.

An analysis of the transcript data for themes allows the following responses to the research questions.

3.1. Research question one

What does the cognitive disequilibrium resulting from discussion about point nine repeated reveal about in-service teachers’ understanding of real numbers?

3.1.1. There exists a teacher who believes that real numbers correspond directly and solely (isomorphically) to our physical experiences or sensory perceptions

Maude has a sense of number that is intertwined with her sense of measurement, as revealed in her explanation of buying three candy bars for a dollar. When Maude is confronted with situations where a measurement system does not have units to express a value – in this case one-third cents – then the value is adjusted to fit the measurement available. For Maude this goes beyond rounding, where one would approximate the number so that it can be expressed in the measurement system. She appears to change the value of the number so that it can be expressed in the measurement system.

Maude’s responses demonstrate that the relationship between point nine repeated and one is important in elementary grades. The researchers saw its usefulness in exposing misconceptions about relationships between repeating decimals and fractions, and Maude’s responses show that it is of practical importance to elementary teachers as well. Maude’s responses reveal that teachers can have difficulty accommodating the practical use of number representations within a commerce system and the mathematics they are learning and teaching in school. Because of her earlier statements about using long division to create representations of repeating decimals, Maude most likely understands that three for a dollar results in one candy bar for 33.33… cents, at least from a purely mathematical sense. But seeing the conflict between the actual value and the coins available to make the purchase, she decides to accommodate the currency system by saying that the one candy bar has a value of thirty-three cents. Instead of saying that the candy bar has a value of 33.33… cents and our currency system lacks the coins to make this purchase, she changes the value of one candy bar. This in turn presents her with a new conflict when adding up the values of three candy bars, which is now 99 cents. This leaves her wondering whether the store’s register is ringing up the correct price.

In reaction to the interviewer’s questions about the relationship between point nine repeated and one, Maude indicates that “to say 99 cents can be rounded up to a dollar” would be confusing to kids. Later, Maude discusses medicinal applications without any acknowledgement that point nine repeated is not used in common medicinal dosing. She goes on to imply that questions around numbers like these are “over-thought,” “distinguish elementary and high school,” and are “off track” when addressed by teachers. In that same response, Maude suggests that she links her understanding of number to whether it is expressible in standard notation in a finite amount of time, saying she tells the kids “for the rest of your life you would be writing threes and you’d never get done” Maude repeats this notion when she gives a faulty explanation of why one third is not actually equal to point three repeating, stating, “if you took it out forever . . . we wouldn’t be there to see the end.”

There is a pattern in Maude’s thinking. When Maude is confronted with a question about repeating decimals, she situates the number in a sensory scenario, and when some of the information about the number doesn’t situate well, Maude adjusts the number to accommodate her sensory perspective. While Maude reveals a great deal about how she accommodates mathematics that does not fit her current conceptions, she also reveals a great deal about the conceptions themselves. Maude gives strong evidence that she does not construe numbers as points on a number line or objects independent of her sensory perceptions of the physical world.

3.1.2. There exist teachers who believe that point three repeated is not equal to one-third

When asked whether one-third and point three repeated are truly equal, Maude states that these numbers are not truly equal and ties her understanding to the number of digits that can be written in a finite amount of time. Toward the end of
this response, she appears to be moving toward an argument that these numbers are in fact equal; however, when asked to plot one-third and point three repeated on the number line, she reveals the opposite. Maude replies that she would plot the two numbers “right on the same line, for this age children,” [She indicated same place on the number line to the interviewer] but would show “older children” that point three repeated is just a “smidgen bigger” than one-third. Illustration 1 displays written work which Maude generated to accompany her explanation.

Maude’s comment suggests that she believes point three repeated and one-third are not located at the same place on the number line, but teaches that younger children that they are.

Later in the interview, Maude gives even more insight into the nature of this misconception. When asked to explain what she means by “close enough,” Maude explained that point three repeated is not equal to one-third, “but it’s close enough for what most people have to use in their everyday lives.” Maude’s misconception seems to hinge not only on a belief in an infinitesimal (when she discusses the “teeny bit missing”), but also on a belief that real numbers exist only as they relate to everyday experiences.

3.1.3. *There exist teachers who believe that “small” quantities are irrelevant, and that rounding and truncating repeating decimals to terminating decimals is a means of making sense of these numbers*

When asked about the relationship between point nine repeating and one, Maude tells the researcher that [the missing amount from point nine repeated] is “such a small portion that it’s irrelevant so you could round up to one.” In response to the multiplication argument, Maude states that the argument “shows point nine nine nine is close to one, and it shows that these numbers [decimals with a single repeating digit] go on infinitely, and it shows that for everyday purposes, you have to just know that these percentages or [non-repeating] decimals are close enough.” When asked to clarify her understanding, Maude affirms her conviction that terminating representations are close enough and situates her understanding in her notions of currency. This conception may differ from the belief in an infinitesimal cited in the literature review. Theoretically, infinitesimal quantities have measure zero. Maude appears to hold the view that quantities of very small measure are inconsequential.

3.1.4. *There exist teachers who believe that one is a whole, it can be expressed in only one way, and its value is explicitly tied to notions of units*

Maude’s responses suggest her understanding of whole numbers is separate from her understanding of decimals and fractions. She discusses showing kids that point nine repeated “is the ultimate number to show kids that even this isn’t one whole.” Other teachers interviewed for this study indicated similar beliefs. Tara, a first-year teacher, discussing her beliefs about point nine repeated and the arguments she has seen, states “I just think of one as it’s a whole.” Jeffery, a fourth-year teacher in his first year teaching mathematics, responds to the interviewer’s questions about the relationship between point nine repeated and one by saying, “If it was equal to it would be one whole unit. . .if it was a whole they’d call it a whole, they call it one.” When asked about decimal representations of whole numbers, all of the teachers limited their responses to augmenting the nonzero digits with zeros in various place values. More research is needed to completely understand the nature of this misconception, but the participants presented evidence that they have difficulty distinguishing the number 1 from one of its labels “whole number” as well as from their notion of units (one whole).

3.1.5. **Teachers’ understanding of real number can be tied to their teaching and perceptions of their students’ learning**

Maude gave responses that suggest her understanding of number is tied to her perceptions of teaching and learning. In response to the interviewer’s first question about the relationship between point nine repeated and one, Maude states “I don’t think it’s equal because, uh, I think that would be confusing to kids.” In response to the addition argument, Maude calls these questions “overthought” and the “difference between elementary and high school teacher.” Participant Tara, a first year teacher, in response to the multiplication argument that point nine repeated is equal to one, stated, “I think it would be really confusing for the kids to have to, um, wrap their head around it because it’s hard for me.” This finding echoes what has been commonly reported: that teachers’ own content knowledge, and the knowledge of content that teachers hold in regard to their students, is tightly linked (Leikin & Levav-Waynberg, 2007).

**Illustration 1.** Maude’s number line diagram, showing that .333... is “just a smidgen bigger” than 1/3, and that .666... is bigger than 2/3, but that the sum of .333... and .666... is less than one.
3.2. Research question two

What types of conflict resolution strategies are invoked by participants when they are confronted with cognitive disequilibrium about the relationship between point nine repeated and one?

3.2.1. There exists a teacher who modifies “truth” in order to fit existing beliefs

When Maude was asked to add point three repeated and point six repeated, she acknowledges that the fractional representations do in fact add to “one whole.” However, when asked to consider their decimal representations, Maude uses a rounding and truncating (to the hundreds place value) approach to accommodate her belief. Illustration 2 shows the written work Maude generated as she spoke.

At the same time, Maude’s adjustment of reality to fit her intuition about repeating decimals is inconsistent. Maude makes point three repeated and point six repeated larger than one-third and two-thirds respectively to generate a sum equal to one. Yet in the written work she generates during her (Illustration 1) Maude has added two numbers greater than one-third and two-thirds respectively and generated a sum less than one. While this might demonstrate misconceptions about number beyond repeating decimals, we contend that when viewed with the rest of the evidence, it illustrates how Maude forces all number situations into schemas that accommodate her inadequate and often incorrect views of numbers.

3.2.2. There exist teachers who will dismiss conflict as unimportant or uninteresting as it relates to their teaching

We examined in Section 3.1.2 how a teacher’s perceptions of students can influence the teacher’s own understanding, but we can also find evidence that teachers use their perceptions of student understanding to dismiss their own cognitive conflict. In response to the addition argument, Maude calls the interviews questions “overthought” and states that even in high school, discussing these concepts would be “off track.” She uses these perceptions to move to what becomes a recurring response, that small portions are irrelevant. She repeats this line of thinking in response to the interviewer’s recap of the multiplication argument, when she calls point nine repeated and one “close enough” for most “every day processes” and later when she refers to higher level science or math. In sum, Maude uses concepts of her students, mathematical “daily-life” applications, and higher mathematics to dismiss the conflicting information as unimportant. Maude was not the only participant to express such a view. In response to the multiplication argument that point nine repeated is equal to one, Tara (the first year teacher) states “I think it would be really confusing for the kids to have…umm wrap their head around because it’s hard for me.”

3.2.3. There exist teachers who create conceptions that contain the means to “work around” cognitive conflict

The transcript reveals multiple instances where Maude attempted to work around the conflict she was presented. This differs from adjusting reality described in Section 3.2.1 because she did not accommodate the conflicting information by changing the mathematics she already claimed to know. Instead, throughout the interview, Maude uses her notions of close enough, rounding, and application problems to say that if a person can just think through this “rationally” the conflicting information just makes sense. The researchers understand that Maude’s comments have been placed in several categories. We contend that Maude’s strategies are complex and multi-dimensional and that she used multiple strategies of conflict resolution over the course of a short interview.

3.3. Research question three

When faced with cognitive disequilibrium about the relationship between point nine repeated and one, what do teachers reveal about misconceptions they hold and ways they might transmit these misconceptions to their students? When placed
in cognitive disequilibrium resulting from discussions about the relationship between .999... and 1, in-service teachers have the potential for teaching less than accurate mathematics. While some of the notions listed in the following section were discussed in earlier sections, we feel it is important to reiterate them here with specific attention to elementary school teaching. Some readers may feel that the relationship between .999... and 1 is not an important topic for elementary school teachers. This section discusses the evidence we found throughout Maude’s responses that disequilibrium around this equality can arise naturally in the context of early grades mathematics instruction and can result in problematic treatment of mathematical topics clearly situated in the curriculum.

3.3.1. Teachers reveal how their belief in the existence of an infinitesimal can be transmitted to students during instruction

When asked about her perspective on the relationship between point nine repeated and one, Maude comments that point nine repeated is the “ultimate number to show kids” that “there’s a teeny bit missing.” Maude’s misconception could arise from one of two misconceptions: a belief in an infinitesimal or a belief in a finite Arabic representation of point nine repeated. Either way, if this was shown to children, student misconceptions could arise, particularly if later in their academic careers (say in Algebra II) the students learn the misconception that point nine repeated “asymptotically” approaches one (Burroughs & Yopp, 2010).

As stated in the literature review, Tall and Schwarzenberger (1978) theorized that if teachers hold misconceptions regarding infinite decimals and sequences, and communicated this unease to students, then this could be one source of the difficulties students have with these related concepts.

3.3.2. Teachers may teach mathematics they indicate they understand to be incorrect

As discussed Section 3.1, when Maude was asked to plot one-third and point three repeated on the number line, Maude states that she would plot the numbers “right on the same line, for this age children but would show “older children” that point three repeated is just a “smidgen bigger” that one-third.

We found also found Maude’s response extremely interesting from a pedagogical perspective as well. Some literature examines the teaching of incorrect mathematics that a teacher believes is correct, but we have seen no literature examining the teaching of correct mathematics that the teacher believes is incorrect. Maude’s comment suggests she believes point three repeated and one-third are not located at the same place on the number line, but teaches younger children that they are. While the information Maude gives these “younger children” is accurate, it may not be delivered with confidence, given that Maude states she doesn’t believe it to be accurate.

3.3.3. Some teachers may teach that real numbers correspond directly and solely to physical experiences

Maude gave responses that suggest her physical view on numbers extends to all real numbers. As discussed in Section 3.2, Maude ties her understanding of repeated decimals to money and time. She situates her understanding in her classroom instruction with comments like “to say 99 cents can be rounded up to a dollar” would be confusing to kids, and “I tell the kids for the rest of your life you would be writing threes and you’d never get done,” and questions around numbers like these are “over-thought,” “distinguish elementary and high school,” and are “off track” when addressed by teachers.

Maude gives an example that shows the practical importance of teacher knowledge about repeating decimals and their sums, when she accommodates the U.S. currency system instead of accommodating the number by saying the currency systems lacks the coins to make an exact purchase. Applications involving three for a dollar are in the scope of elementary teaching, are in the practical experience of elementary school children and are certainly not “off track.” Scenarios such as Maude describes can easily arise in elementary mathematics classrooms. There is a potential for students to adopt their teachers’ misconceptions if a teacher gives an explanation similar to Maude’s rather than a discussion of real numbers (or all numbers on the number line) and the set of numbers expressible in the U.S. currency system.

3.3.4. Some teachers may teach that approximations are good enough and “small” doesn’t matter

Maude asserts that small portions “after ten thousand places” are good enough. She expressed these thoughts again in (“it shows that for everyday purposes, you have to just know that these percentages or decimals are close enough”). She does conclude this comment with the caveat “for most every day processes”; however, she uses this approach in resolving her difficulty with the mathematical process of addition. The approach is also present in her number line placement demonstrated in Illustration 1. In these situations there is no physical reference; these are real number notions not associated with any application problem, yet she rounds and truncates to resolve the conflict without reference to the numerical information lost with such an approach.

This phenomenon is not unique to participants in this research. Tall (1994) hypothesized that a student may see no harm in approximating an irrational number with a finite decimal. Here we have evidence that this exists among teachers even for rational numbers in the form of repeating decimals that have well-known fractional equivalents.

Considering these three cases together, what emerges? Teachers’ interactions with the curriculum do not cement correct understanding. Thus, properly addressing repeating decimal concepts is important before calculus.

The researchers have no evidence that Maude actually teaches these misconceptions. No data of what is enacted in Maude’s classroom was collected. Her comments may have been influenced by the disequilibrium the interview produced. On the other hand, Maude’s comments illustrate how a teacher’s inadequate understanding of repeated decimals can result
in less than accurate reflections on mathematics that is within the scope of fifth grade curriculum, as well as result in less than accurate reflections on student learning. Maude’s comments give clear warrant for more careful attention to repeating decimals in elementary school preparation and professional development.

4. Conclusion

This research demonstrates that some elementary teachers have incorrect views of decimals with single-digit repetends and that cognitive disequilibrium about the relationship between the decimal point nine repeated and one reveals misconceptions about repeated decimals in general. This research differs from previous research because it examines the views of in-service elementary school teachers and because it demonstrates how these views can result in problematic instruction at the elementary grades.

Our purpose in conducting this qualitative research is to demonstrate what variation exists in teachers’ conceptions of repeating decimal concepts and to provide evidence about the nature of teachers’ misunderstandings. We have been careful in the presentation and analysis of these interviews not to generalize or to assume that the phenomena revealed by our participants are common. The evidence from these interviews showing that these phenomena do exist indicates this problem is worth study on a larger scale. From this research, we know that misunderstandings about concepts related to the concept of infinity arise well before the formal study of calculus and can impact the teaching of the elementary school mathematics curriculum. Of particular interest is that teachers question the equality of 1/3 and .333…, an equality that is firmly rooted in elementary mathematics content.

While this research did not examine strategies for alleviating teachers’ misconceptions about decimals with single-digit repetends, it is important to reflect on such strategies for future study. Certainly the kinds of arguments we provided to the teachers are not convincing to all. One participant, Jeffery, the fourth-year teacher in his first year of teaching mathematics, provided an example of a teacher who was convinced by the arguments in the interview. At the end of his interview, Jeffery responded he was most convinced by the addition argument. In his own words, this was evidence, rather than proof, that .999… is equal to 1. Of particular interest is that the number line argument, which comes the closest to explaining why the numbers are equal, is not convincing to all the teachers. This may be symptomatic of how robust their belief in infinitesimals is, or it may stem from their view of repeating decimals as processes and not numbers.

If it is the case that some elementary teachers do not truly believe that decimals with single-digit repetends are numbers rather than processes, then APOS theory (Asiala et al., 1996) may offer a strategy for developing a more mathematically correct and complete concept image. Briefly, APOS theory explains that mathematical understanding comes through structures called actions, processes, objects and schemas. APOS theory says that a mathematical concept “begins to form as one applies a transformation on objects to obtain other objects.” A transformation is an action when it requires “specific instruction as well as the ability to perform each step of the transformation.” After repeating and reflecting on an action, it becomes a process, “a mental structure that performs the same operation as the action . . . wholly in the mind of the individual.” When the process becomes a totality, then the process becomes a “cognitive object.” Finally, objects that are organized and linked become a schema (Dubinsky et al., 2005, pp. 338–339).

In our interpretation of APOS theory, decimals with single-digit repetends become objects when teachers operate on them. Even though repeating decimals are generated and discussed in school mathematics, opportunities to use repeating decimals as a number or an object (e.g. perform computations with them) appear to be scant or non-existent. From this perspective it is no wonder that some elementary school teachers have incomplete and problematic views of these numbers.

Whether an individual instructor feels that the equality of point nine repeating and one is an important topic for elementary teacher education or professional development, the equality of point three repeating and one-third is. This topic is in many fifth grade curricula and is explicitly taught. Often, the equality is generated using the standard long division algorithm. By examining these three transcripts, the evidence is strong that without a solid understanding of repeating decimal concepts, teachers have the potential of undermining student understanding of important concepts in the elementary school curriculum.

Appendix A. Interview protocol

Some background information: What math classes did you take in college? Any graduate work? In high school?

We’re really interested in finding out how people describe different relationships between numbers. People use lots of different ways to explain the connections and we’re really interested to hear how you describe them.

A1. What are the relationships between decimals and fractions?
   1. If none mentioned ask about 1/3?

A4. What are the relationships between decimals and whole numbers?
   1. If none mentioned ask about 2.07?

A5. What are the relationships between fractions and whole numbers?
   1. If none mentioned ask about 2/17?

A6. What is the relationship between multiplication and division?
We're interested in how you think about multiplication and division its relationship to forming decimal representations

7. Can you write 6 ÷ 2 = 3 as a multiplication problem?
8. Consider the division problem [show the long division problem showing 125 divided by 5 is 25];
   Can you describe this as a multiplication problem?
9. Can you give an example of a fraction that has a repeating decimal equivalent?
10. What is the relationship between 1/3 and .333...?
    1. How would you describe and explain this relationship?
    2. Can you show me?
    3. Can you use long division to show me?
   Some people think that 1 is equal to .9 repeating and some people don't. I'd like to hear what you think about it and why?
11. Earlier you showed how multiplication and long division are related, and then you showed how long division connects _,_ and _. (Show teacher their earlier responses in hard copy.) Use the same process to rewrite the long division of 1/3 as multiplication.
12. Please show me how you would multiply 3 and .333...?
13. Does or doesn't this show that .999... = 1?
14. What is 1/3 as a repeating decimal?
15. What is 2/3 as a repeating decimal? (If .67: is that your exact answer or did you approximate?
16. Add 1/3 + 2/3. Add their decimal and fraction form. What does that show?
17. Is this convincing? If not, why not?
18. Is either of the arguments (addition or multiplication) more convincing? Why?
19. If either of these don't show that .9 repeating = 1, what step wasn't valid?
20. Where would you plot .9 repeating on the number line?

Appendix B. Full transcript of Maude's interview

**Interviewer:** We're really interested in finding out how people describe different relationships between numbers. People use lots of different ways to explain the connections and we're really interested to hear how you describe them. What are the relationships between decimals and fractions?

**Maude:** How would I teach that? [Interviewer: Either one. How you understand it, or how you would teach it] Ok. Um. I always think of decimals and fractions as the same thing but used in different ways. And so I always talk about how it is important to know about 5 basic fraction equivalencies to decimals: a half, a quarter, a third, two thirds, you know, those basic five, probably, um, three fourths. Whenever I start fractions I immediately start showing equivalences to decimals, so that kids can see immediately and start memorizing the sameness of the two. And I, uh, I also try really hard to show percentiles at the same time because then we can talk a little bit about how they're different than decimals but yet the same, and only used out of a hundred. So that's kind of how I start. And that's always how I think of them too. When I see a third off at the store I immediately think 33 percent or vice versa. So I have to pay 67 percent.

**Interviewer:** So what about relationships between decimals and whole numbers?

**Maude:** Part of a whole. I start out just saying decimals are part of a whole. That's how I think about them, um. And then a lot of our resources we've always taught from talk about decimals numbers even including whole numbers. So, you know, one and seven tenths is called a decimal number. And so I talk about whole numbers and decimal numbers a lot of times. I personally talk about decimal numbers as less than a whole. And then later, after they start getting the hang of that, then I talk about how a decimal number could have a whole number and a part of a whole.

**Interviewer:** What about a number like 2? Do you talk about that whole number? How do you...

**Maude:** When we get into fractions, especially when we get into fractions, we talk about how. Let's go back to one. Can I talk about one? We talk about how one is the same as four fourths, eight eighths, ten tenths, a hundred hundredths, a billion billionths. And we talk about those pieces, you know, however big the piece is cut into or the pizza is cut into, if you have all the pieces, it's one whole pizza. And, so, um. You know what's happening, is I'm forgetting the question as I start rambling.

**Interviewer:** The whole number,

**Maude:** Oh, the two number. So I start with 1. I guess, As far as the number 2, by fifth grade, I have to say I think the kids have a pretty good concept of 2. And it's only when we get into sixteen eighths that I might have to draw the picture and say, see how this is two full pies, divide the bottom into the top, eight into sixteen is two. It means the pies are cut into eight pieces. All eight pieces are in both of them. There are 16 pizzas, pieces of pizza, and that's the same as 2. So I might get into that later in fractions.

**Interviewer:** I always say, I always talk about how any whole number can have a decimal hidden behind it, sometimes they just leave it invisible because they don't need it. As soon as you need it, it can pop up or you can put it there.

**Interviewer:** Following up, we said decimals and fractions, and we just did decimals and whole numbers. So what about relationships between fractions and whole numbers? I think that's what you were describing,
Maude: I probably would. Then I would talk about improper fractions, you know, when we get into that then I would show how, and draw and let them manipulate how that's greater than one, but you can still have more of the pieces. Now that's what I wanted to show you in the new math program. [Goes to get the textbook.] They start out immediately going to improper fractions. I've been in the program for three weeks and they haven't talked about mixed numbers yet. And I'm not familiar enough with the program to see where it's going to take us. And when it's going to change that improper fraction into, um, more. But they continually say, Circle the fractions that are greater than one. That's how they start. Circle the fractions that are greater than one. So they show pictures. Or have a number line divided into pieces. For example if it's six fourths, they talk about the unit is cut into this many pieces. Four pieces. How many are shaded? 6 are shaded. Circle it if it is greater than one. So that's where they're starting. So I thought that was interesting. I immediately have always, 20 years, have always reduced to lowest terms immediately. And this book doesn't. It's called Connecting Math. It's a remediation program, you pretest kids, there are several levels. I know in our building we start as low as C, which is probably about a second grade level. This I would call about a third grade level. This is what I'm teaching now, is D. And um, it's all scripted. All scripted. You know, you say everything and they repeat phrases after you. Um, it's really hard to stick to after you've taught so long. Because you just, all of a sudden, you feel like oh I'm losing 'em, if I have to say this one more time, you know, I might lose them. And so I've not been, I've really, I've not followed the program as religiously as I should simply because I just feel like. What we did was we restructured our math groups three weeks ago and so lot of of these kids feel like they're in the low group, and they are, but they've never been told that, but they don't have to be told that. And so some of them are thinking right away this is way too easy, because it is, and we've been adding fractions with different denominators, and now we are taking them back to this, and it's just less than one or more than one. Counting the pieces. I think it's going to pay off because they're going to get a much more concrete idea of what a fraction is than they had, according to the pre test we assessed them on.

Interviewer: What about um whole numbers like this, how do you approach that, five over one.

Maude: What I would do is, I usually in my old program I wouldn't bring this up. Until we got to a point where we were multiplying or dividing fractions. But this program starts showing them right away, not right away, we haven't done it yet, but by next week it'll be showing you don't have to cut the whole up and you can still make a fraction out of that. And so if you have five whole units, and that's what this book calls them always, units, whereas I'm using pizza, pie, candy bars. It's five whole units is the same as five over one whole unit, or the unit's not cut up, it is one whole piece. But. So. It's very scripted. And you know maybe for a beginning teacher it would be more guided. If they were unsure of themselves in math it would certainly be a safe way to go.

Interviewer: What about, you mentioned a little bit about this, the relationship between multiplication and division? Not necessarily limited to fractions and decimals. But in general, how do you approach that?

Maude: Ummm. Multiplying. I always, you know, I talk about, umm, repeated groups of the same number or repeated groups that are the same. Adding the same number, a short way to do it is multiply. Umm. I talk about either sets, you know, sets of three or, umm, I can't think of how I say it. But sometimes its sets of three, you know, three clothespins in each set. Or sometimes it's, uh, five pieces cut into the pie, three of those, so five times three. So units cut into pieces or sets of things, and then for dividing I try really hard to talk about it in an opposite way. Now we have all these pieces of pie and I really want to take all the leftovers and see how many pie pans I need. You know, Thanksgiving's over, I really don't want all these dirty dishes lying around, so let's divide and see how many sets of seven I can get so I can fill each pan with seven pieces to make a whole pie again. So that's often how I talk about it. Or let's divide the sets. Okay, we have all these m & m's, and there are twenty of you. And let's divide these evenly. You know. So we can pass them out, one for you, one for you, one for you; or we can divide them into equal sets. Sometimes we have remainders, and that means I get the leftovers, I always say. I try to keep em separate, I always try to refer to them as opposites. Not when we get to fractions though. Yeah.

Interviewer: Ok. So now a few things that I could get you to write. How would you write 6 - 2 = 3 as a multiplication problem?

Maude: Mmm. Six divided by two equal three. As a multiplication problem. Mmm I might have to think about that a minute. Okay. [She's writing, INTERVIEWER reads "Three times two equals six." I guess if you say it that way, that's probably, I'd put three times two equals six.

Interviewer: And then what about the division problem:

\[
\begin{array}{c}
5 \\
\hline
125 \\
10 \\
25 \\
25 \\
0 \\
\end{array}
\]

How would you rewrite that one as a multiplication problem?

Maude: I would probably do it two ways. I would show them that I could do 25 times five. Or these numbers can be mixed up and you'll get the same answer just like in addition and so I could always put it as five times twenty five. Then I would talk about what that means. This is 25 sets of 5. This might be 5 sets of 25. 25 piles of M&Ms with five M&Ms each, five piles of M&Ms with 25 M&Ms each. That's how they're different, but in the end they have the same number of M&Ms.

Interviewer: And for your students, do you name that property? Do you talk about it as commutative?

Maude: We do. We go over commutative associative, zero property of addition. One property, we do go over those, they're in our book that we use in our mainstream class.
Interviewer: Can you give an example of a fraction that has a repeating decimal equivalent?
Maudie: one third, two thirds.
Interviewer: And those are in your, you mentioned you have that set of five you tell them they have to know.
Maudie: I would do one third, one half, 3/2 is and I would probably do somehow a set of one. So that I could show them that one whole is one hundred percent is one point zero zero, in a decimal, or one, and there's always a decimal here. So these are the main ones. If I feel like my class can handle it I'll do the fifths. Umm, I'll go on and I'll do one fifth, two fifths, three fifths, four fifths. Umm, sixths, um, I'm trying to think if, no, sixths, by fifth grade, probably sixths not very often. Tenths are easy. But these would probably be the mains for sure. And 7/4 I forget.

Interviewer: So, if we just focus on one third. How would you describe the relationship between 1/3 and 3.
Maudie: I try to start early when I teach fractions that this line also says equal to, this symbol says times or multiply by. And so I try to get that terminology right away, to say this has a word that goes with it. This is greater than. This symbol is less than. And so I have a couple things. You know I might start out saying, one over three is the same as one divided by three. And this would be, you know, by then we would have divided with decimals, okay, and talked about you can add a zero after the decimal until the cows come home, that's how I always say it. And we would do this, you know, we would divide 3 into 10 is 3, and we would go and do that for a long time. To show them that three goes on forever and ever. And another thing I would do is I would say, when I talk about one out of three is the same as thirty three and a third percent, or thirty three point three percent, or point thirty three percent, this is percent, zero point three percent. Um, I can't remember a fifth grade I've ever had that shows the repeated sign. But sometimes I throw it in anyway. You know, just because, again, and a lot of times it depends on the class you have. You don't want to confuse 'em by giving em too much information. And I think our new math books are kind 'a bad that way. I think our new math books try to show all the different ways you can solve this problem. And sometimes, when kids are first learning a concept, I think it's a little better just to show them one. And maybe one percent being so as a teacher, have them get really comfortable with it. Because fifth graders aren't always developmentally ready to take on 2 or 3 different ways to solve the same problem. You think it is giving them a choice or giving them a gift, but sometimes it is just giving them confusion, you know, don't you think? So if I want, Did I get you the answer?

Interviewer: Yes. You mentioned the ellipses, the 3 dots to show it goes on forever, do you use the bar notation?
Maudie: Yes. I've used that before as well.
Interviewer: If you were to write it down for your students, how point three repeating is equal to one third, what would they look at on the board?
Maudie: They would see zero point three three is equal to one third, and then when I went to zero point six, I would round it. I would round it to 0.67 for two thirds. I think that probably I habitually put the three dots after, what did you say it is called?

Interviewer: Ellipses, so you would prefer that over the bar?
Maudie: I probably use that more than the bar. But I think my friend [Teacher's name] next door uses the bar, I do. Because I've seen it on her board before. And maybe one more place past the decimal point, then.

Interviewer: And how do they see it in their textbook? Have you gotten to this yet this year?
Maudie: No, we do, we've gotten to it, but I don't know that they think about it. Let's check. (Looks in book) Is this a publisher you've seen? Scott Foresman. Comparing decimals. I'm trying to think where it might be most likely. Addition, comparing, percent of a number, decimals and fractions, is that what you're going for. That's showing percents. [finds that the book uses .331/3,]
Yup that would be the first time they would see that. I show it way earlier. I show it as soon as I introduce fractions. Which is more of a different math program. You know how you pick what you like. Um, let's see, I won't look long. But just, I don't use the whole program, I have parts of it, oh here it is. This investigations, it uses a lot more whole thought, whole process thoughts like that. And so when you get into investigations and you're doing name the portion, you can see I have pages sticking out of it because I use it more likely than anything, of this set this is what I use it for, it does that it immediately, compares all the way across the board. And different ways to look at it. Let's see if they have some repetitiveness numbers in there that they are showing.

These are all different ways to say fifty percent. I'll just go and if you see anything you stop. The problem with this as a whole program, it does lots of hands on, lots of investigating, thinking, doesn't do a lot of practice. So, if you like homework. See it had… [Interviewer: they're just placing it on a number line]. Well anyway, it was another resource. Interpreting decimals. They're pretty much taking everything three places past the decimal, they're not much showing what they do if they repeat.

And let's go on and I'll just keep looking while we're talking.

Interviewer: You don't have to, I was just curious.
Maudie: Oh, here we go, just when we give up, just three places past the decimal and pretty much leave it at that I guess.

Interviewer: Here's a question to make you think. Some people think that 1 is equal to .9 repeating and some people didn't. I'd like to hear what you think about it and why.
Maudie: I don't think it's equal. I don't think its equal because, uh, I think that would be confusing to kids, to say that 99 cents can be rounded up to a dollar. You know, just right to money is immediately what I would think of. Umm. But definitely when you get to more scientific things I can see where that might be seen that way, you know, in different high level sciences, possibly, but, even then you think they'd want to be more particular about the size or the number. So I don't ever think of point nine nine nine being one. I always think of it as less than one. I think it's the ultimate number to show kids that even this isn't one whole. There's a teeny bit missing. This is good for me because I never think I'm very opinionated. And then, whoa!

Interviewer: A few questions related to that. Before you showed (long division) that multiplication and long division are related and in #11 you showed how long division connects 1/3 and 3. (Show her earlier responses in hard copy.) Use #8 to write division as multiplication.

Maudie: Umm, okay so if I was going to show that. I would probably use the word of. And I would talk about doing thirty three hundredths of one whole. Now, Well that would be one way of doing it. That would be one of the two ways. And then I would talk about how the word of, when you're finding parts of a whole, that's the same as multiplication. Now if I wanted to take the whole number three of the one third and say three of one. Now see that doesn't look like that would be the easiest way to describe it to children. And so, I might try this, I would probably try this, because this is something real. Sometimes you need to find thirty three percent. Or thirty three hundreds of a dollar to find out how much it costs if it's on sale by taking a third off.

Three of one gives me a whole different vision. Three of one gives me, you know, I want to take one pie, and or, I don't know, I wouldn't take three times one. Probably. Well, it wouldn't work anyway. Oh I need three times point three, though, really, don't I. Ok, I'm sorry. So if we said. I would probably do it like that. I'm sorry. Thirty three percent or thirty three hundredths of three, is equal to one, but it's not going to come out evenly. And so that is where our point nine nine nine nine comes in doesn't it.
Interviewer: So that is my next question. Show me how you would multiply three times point three repeating.
Maude: You get point nine nine nine nine nine. So then I would have to talk about how umm, rounding comes into the equation. How you have to round up, round down sometimes. You know and with the zero point six six six seven, times, one, you know you've rounded this up. This, you have to remember that you have a number that you're not accounting for, that three that goes on forever. You can't round this up to a four. This is only a three and so. You have a little sort of missing portion in two thirds and one third that when you want to multiply to check it doesn't work so well. Yeah, but I still would not, I still don't think I would talk to kids about how it's like one. But maybe eventually you would get to that, because you could say Yeah, Point nine nine nine nine nine forever rounds up to one. And there you have your whole dollar. I must never have had to explain that. Because I realize now I'm struggling with how I would say that.

Interviewer: If you were to say which one of these is exact and which has been rounded. This, the three times point three three equals one, you'd say this one is the equality? But it's in the multiplying is where you're missing a piece? If this one tells me that three times point three repeated equals one. But this one tells me that three times point three repeated is point nine repeated, which one is the true equality and which one has the rounding.
Maude: Well, you never get done with the division. It goes on forever. So unless you want to go on forever the rest of your life dividing that, you better stop, and round it. And when you multiply point three three three repeating times three to check your work, you have to use logical thinking to see that the reason it doesn't come out as one is because you never got the true number on the top. Right? It's as close as I could come.

Interviewer: Do you think this shows that point nine repeated forever is equal to one, or it doesn't.
Maude: No, I still don't.

Interviewer: I'll try another argument you might see. If I have the fractional representation of 1/3 and it's equal to its decimal representation, so, point three three, and I use your dots, and I have the fractional representation of 2/3 and its decimal representation, point six six, and I'll write it like that. So if I add the fractional representation, Maude: You'll get one whole. Three thirds is one whole.

Interviewer: And their decimal representations.
Maude: I have to admit, I have to admit, I would have rounded these then. I would have rounded, I would have kept the thirty three thirty three. And I would have rounded this to sixty seven and I then would have gotten one whole.

Interviewer: Ok, but if I leave it in the exact form, which is the point six six.
Maude: Yeah. Then you're going to get point nine nine nine nine again.

Interviewer: Does that show point nine repeated is equal to one, or it doesn't.

Maude: No. I guess I still don't. I think it just, I think you shouldn't be able to take a number like point nine nine nine nine forever and turn it into one. Because what if you need nine nine nine nine hundredths of a medicine, and you give a hundred, or you give 1.0 milligrams of medicine, and it's too much.

Interviewer: But I think questions like these are overthought, which is probably really the difference between an elementary school and a high school teacher. Because we don't come up against theories like this being questioned like this like. In high school the kids would love to get the teacher off track with that theory, I can see that.

Maude: I guess eventually you might have to cave in and say that point nine nine nine nine equals one, but only for the fact that you don't have the rest of your life, you know, to add all those repeated numbers. Which is what I always tell the kids, for the rest of your life you would be writing threes and you'd never get done. And it's such a small portion, you know you could say that, after ten thousand places or so, it's such a small portion that it's irrelevant so you could round up to one. I suppose you could look at it that way.

Interviewer: Okay, so, is this a true equality then. One third equals point three repeating?
Maude: Well then you could say Not true, because it never comes out even. It never ends, right, I mean, we always talk about those being equal. But uh, well, again, yeah, if you took it out forever then it would be. But we wouldn't be there to see the end, you know I mean. But yah, that would be a true equality, but when we round it then, no, it's not a true equality any more.

Interviewer: So if we do a number line, and place zero one two. If we could measure it exactly, where would you place a third on the number line?

Maude: Okay, Okay, so if I have one third and two thirds, you know, it would be pretty easy to say, I would tell the kids that's the same as point three three repeating, point six six repeating.

Interviewer: So that's where you would put point three three repeating, right on top? Exactly on that spot?

Maude: I would. I would put it right on the same number of line, for this age children. But, not if I was teaching older children. I might show that that [the decimal, shows the written plot] is just a smidgen bigger. But then yah, if you have that [the decimal] a smidgen bigger and that 2/3 [as a decimal] a smidgen bigger, when you turn it to a decimal, then it's finally going to get up to one. I see that point.

Interviewer: Okay, so where would you plot point nine repeated on that?

Maude: I would. To show that I don't believe it's one? (laughing)

Interviewer: Well. Where should it go logically?

Maude: I'd put it as close as I could to one without being on the one.

Interviewer: Even if I could write nines forever. For every nine you give me, I can write one more?

Maude: I guess I wouldn't. Yah, I guess I wouldn't. Should I be? Are you trying to tell me I should be? [Interviewer: I don't know, these are the questions.] I still would not.

Interviewer: Just to recap. We talked about three different equivalencies. One was this long division, and how it relates to this multiplication, and that you say comes close to showing that point nine repeated is equal to one? Or it shows that point nine repeated is close to one?

Maude: It shows point nine nine nine is close to one and it shows that these numbers go on infinitely and it shows that for everyday purposes, you have to just know that these percentages or decimals are close enough. For most every day processes.

Interviewer: They're close enough. Right, and if you, and I always talk if you're in a higher level field of science or math, you might have to break that down more. But I still don't think I would have ever said that point nine nine nine is equal to one.

Interviewer: Even though you would say that point three 3 3 3 repeated is equal to one third? But this is what you're saying, it's close enough, or this is truly equal?

Maude: It's close enough, it's not true equal, but it's close enough for what most people have to use in their everyday lives. Yep it's close enough. It doesn't. When you buy something, you know it does still catch me. When I see cash registers process things like that though where it does end up really coming down to ninety nine, you know like three candy bars for a dollar and you're like should it really be a dollar. You know, or is it going to ring up as 99 cents?
References


